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A thermal boundary layer in a reversing flow

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Conventional boundary-layer theory cannot be applied when the fluid velocity outside the layer changes direction, and the leading edge of a finite body changes ends. In this paper an approximate method for examining the details of the boundary layer during a single flow reversal (occurring at $t = 0$) is described. It is based on the expectation that (*a*) long before reversal ($t < -t_1$), there will be a quasi-steady boundary layer appropriate to flow in one direction; (*b*) long after reversal ($t > t_2$) there will be a quasi-steady boundary layer appropriate to flow in the opposite direction, and (*c*) in between there will be a period of pure diffusion. The method is applied to a simple heat-transfer problem, in which a fluid of thermal diffusivity D flows with uniform velocity $U = At$ over the plane $y = 0$; the strip $0 < x < L$ of the plane is maintained at temperature T_1, while the rest of the plane and the fluid far away have temperature T_0. The approximate solution is compared with an exact solution of the boundary-layer equation, and is shown to give an accurate prediction of the heat transfer as a function of time. The boundary-layer approximation itself breaks down in regions of length $O(D^{\frac{2}{3}}A^{-\frac{1}{3}})$ near the ends of the heated strip, as usual; it also breaks down in the neighbourhood of the point $x = \frac{1}{2}At^2$, $t > 0$, at which the influence of the new leading edge is first felt after flow reversal. In a solution of the full equation, this region is examined in detail, and boundary-layer theory is shown to be sufficiently accurate for the calculation of heat transfer.

1. Introduction

The use of boundary-layer theory simplifies the analysis of steady flow past a solid body largely because the governing equations become parabolic in x, the co-ordinate measured along the body surface. In other words the velocity profile at a given value of x depends only on the upstream flow, at smaller values of x, and is independent of conditions downstream. This simplification can also be made in an unsteady flow, as long as the velocity of the fluid far from the body is always in the same direction. If it reverses its direction, however, the 'leading edge' of the body changes ends, and 'upstream' does not always refer to smaller values of x. The velocity profile at a given value of x will depend, at different times, on the flow at both larger and smaller values, and even if the boundary layer on the body surface remains thin all the time, the simplifications which that usually permits are lost.

If the direction of flow reverses repeatedly, with a high frequency and a small amplitude, the flow is dominated by an oscillatory Stokes layer on the body surface, and all the details can be analysed accordingly (see Riley 1967). Such an analysis, however, requires that the two parameters $\nu/\omega L^2$, and $U/\omega L$ are both small, where U is a typical oscillatory velocity, L is the length of the body, ω is the angular frequency of the oscillation and ν is the kinematic viscosity of the fluid. This paper represents a first step towards the analysis of boundary layers in flows which reverse their direction with a low frequency and possibly a large amplitude, so that neither of the above parameters is necessarily small. The object of the paper is to propose an approximate method for performing the analysis, and to check the accuracy of the method by an exact solution of a particularly simple problem.

Perhaps the most important example of a flow which repeatedly reverses its direction is that of blood in large arteries. In a normal human aorta, for instance, the blood velocity on the centre-line oscillates periodically (but not sinusoidally) between a maximum value of about $1{\cdot}4\,\mathrm{ms}^{-1}$ and a minimum of about $-0{\cdot}4\,\mathrm{ms}^{-1}$, with a mean value of about $0{\cdot}3\,\mathrm{ms}^{-1}$ and a frequency of 1·2 beats per second. In recent years, there have been a number of measurements of blood velocities in large arteries by means of hot-film anemometry (see Nerem, Seed & Wood 1972). The anemometer is calibrated for *in vivo* studies by placing the probe in a sequence of known steady flows, and assuming that its response in an unsteady flow is quasi-steady. Laboratory studies in known oscillatory flows have shown that this is a good assumption for the probes used, except for a short period around the time when the flow reverses its direction and the blood velocity is low (Seed & Wood 1970). Indeed, these experiments indicate that the quasi-steady assumption is rather better than has been predicted theoretically (Pedley 1972). However, that theory is not applicable if either the blood velocity far from the probe, or the shear rate in the blood at the location of the hot film, reverses its direction or becomes very small. Since such reversals repeatedly take place both in arteries and in the laboratory experiments, the practical value of the theory is limited. It is hoped that the present work will lead to a more universally applicable theory.

Consider a fluid flowing past a flat plate of length L, with a velocity $U(t)$ far from the plate which varies with characteristic frequency ω. Suppose that the Reynolds number $LU(t)/\nu$ is large, and the frequency parameter $\omega L/U(t)$ is small, at all times except for short periods while flow reversal is taking place and $U(t)$ passes through zero. It is therefore plausible that, except during those periods, the convective inertia terms in the equation of motion are large compared with the unsteady inertia terms, and a quasi-steady boundary layer is formed. For simplicity, we shall consider a single flow reversal, and suppose it to occur at time $t = 0$, with $U(t)$ being negative for $t < 0$ and positive for $t > 0$. The first stage of the approximate theory is then to assert that there is a quasi-steady boundary layer for sufficiently large negative and positive values of $t\,(t < -t_1$ and $t > +t_2$, say). The leading edge, from which the boundary layer grows, is not at the same end of the plate for $t > t_2$, after reversal, as it is for $t < -t_1$, before reversal. The next stage is to estimate the nature of the flow for intermediate times. This estimate is based on the recognition that for times close to $t = 0$ the convective inertia terms will be small, since $U(t)$ is small, and will probably be dominated by the unsteady inertia terms. Thus, instead of the quasi-steady balance between convective inertia and viscous terms, there is a quasi-diffusive balance between unsteady inertia and viscous terms. Furthermore the boundary-layer approximation should still be applicable during this interval, as long as there is not enough time for the boundary layer to grow so much that its thickness becomes comparable with L. The approximate theory therefore asserts that, when $-t_1 < t < t_2$, the boundary-layer approximation can still be applied, and the flow is given by the quasi-diffusive balance between unsteady inertia and lateral diffusion. The only important problem now remaining is to identify the change-over times $-t_1$ and $+t_2$. In order to do so, it is necessary to consider a specific example in detail.

The example we choose is one for which an exact solution can easily be found, and used to check the approximate theory. It is that of the thermal boundary layer in a known flow over a heated region of a plane boundary. Since the flow is known, the governing equation for the temperature is linear, and the problem is much simpler than that of the corresponding viscous boundary layer. This problem was suggested by the application to hot-film anemometers, but we make it simpler still by supposing the velocity over the heated region to be uniform ($u = U(t)$, $v = 0$), and not sheared as it would be over a real rigid boundary. The only direct application of the present solution might be to heat transfer in liquid metals, where viscous boundary layers are much thinner than thermal boundary layers because the Prandtl number is small. The problem is not directly applicable to blood flow, but should be regarded as a simple prototype; the approximate method developed here will be applied in a subsequent paper to realistic thermal and viscous boundary layers.

The problem is formulated mathematically in the next section. The approximate solution, including estimates of the change-over times $-t_1$ and $+t_2$, is presented in §3. An exact solution of the boundary-layer equation is given in §4, where it is shown that the approximate theory does give a satisfactory estimate of heat transfer from the heated region. Finally, in §5, the exact solution of the full equation is examined, because the boundary-layer approximation appears to break down near the transition from the diffusive solution to the quasi-steady solution at time $t = t_2$. The detailed structure of the solution in the neighbourhood of this transition is determined, and it turns out that the error in the heat-transfer function as calculated by means of boundary-layer theory is indeed small.

2. Formulation of the simple problem

Fluid of constant thermal diffusivity D occupies the semi-infinite domain $y > 0$, and flows with uniform but unsteady velocity $U(t)$ in the x direction. The region $0 < x < L$ of the plane $y = 0$ is raised to the constant temperature T_1 while the fluid far from this plane has temperature T_0. We wish to calculate the two-dimensional temperature field over the heated region, and in particular (with hot-film anemometers in mind) we seek the rate of heat transfer from the heated region, as a function of time. It is possible to solve this problem, under the boundary-layer approximation, for an arbitrary function of time $U(t)$. However, it is more instructive to examine a single flow reversal, occurring at time $t = 0$; as long as $U(t)$ is sufficiently slowly varying it can therefore be represented near $t = 0$ by a constant acceleration A. The condition for this to be a good approximation is that the time scale for changes in the velocity, say $1/\omega$, should be large compared with the time, spanning flow reversal, during which the temperature field is not approximately quasi-steady; this turns out to be proportional to $L^{\frac{1}{2}}/(AD)^{\frac{1}{3}}$. We therefore require

$$\omega L^{\frac{1}{2}}/(AD)^{\frac{1}{3}} \ll 1,$$

and from now on shall restrict our attention to the velocity field

$$u = U(t) \equiv At, \tag{2.1}$$

where A is a positive constant. Thus the flow is in the negative x direction for $t < 0$, and in the positive x direction for $t > 0$.

We make x and y dimensionless with respect to the length scale $(D^2/A)^{\frac{1}{3}}$, and we make t dimensionless with respect to the time scale $(D/A^2)^{\frac{1}{3}}$, denoting dimensionless variables by the same symbols as their dimensional counterparts. The non-dimensional velocity is equal to t. The governing equation for the dimensionless temperature $\theta = (T-T_0)/(T_1-T_0)$ is

$$\theta_t + t\theta_x = \theta_{yy} + \theta_{xx}. \tag{2.2}$$

The boundary conditions on θ are

$$\left.\begin{array}{llll} \theta \to 0 & \text{as} \quad y \to \infty & \text{or} & |x| \to \infty \\ \theta = 1 & \text{on} \quad y = 0, & 0 & \leqslant x \leqslant l, \end{array}\right\} \tag{2.3a}$$

and

where $l = L(A/D^2)^{\frac{1}{3}}$ is the dimensionless length of the heated strip. The obvious condition to impose on the rest of the plane $y = 0$ is that it is insulated, i.e. $\theta_y = 0$. However, this condition leads to a mixed boundary-value problem, whose solution requires complicated analytical methods like the Wiener–Hopf technique. We therefore replace it by the much simpler isothermal condition $\theta = 0$. In steady problems it is known that this makes no difference to the boundary-layer solution for θ in the range $0 < x < l$. The only difference in the exact solution occurs in small regions, of length $O(1)$, in the neighbourhood of $x = 0$ and $x = l$, where the boundary-layer approximation breaks down anyway. There is no reason to suppose that the changed boundary condition has any greater effect in the unsteady case. The additional boundary condition is therefore taken to be

$$\theta = 0 \quad \text{on} \quad y = 0, \quad x < 0, \quad x > l. \tag{2.3b}$$

As we have indicated already, we are supposing that the region in which θ is significantly different from zero consists of a thin thermal boundary layer in the region $0 < x < l$, together with thin thermal wakes outside this region. We therefore make the boundary-layer approximation, neglecting the term θ_{xx} in (2.2), which is replaced by

$$\theta_t + t\theta_x = \theta_{yy}. \tag{2.4}$$

The condition that this should be a good approximation over most of the heated region in steady flow is that the Péclet number LU/D should be large. However, in this case U passes through zero, so the Péclet number is not always large. It seems plausible that the corresponding quantity in these circumstances should be the dimensionless number

$$LA \times (\text{time scale})/D = L(A/D^2)^{\frac{1}{3}},$$

which is just the dimensionless length l of the heated region. If $l \gg 1$, therefore, we expect the boundary-layer approximation to be accurate almost everywhere; i.e. for all values of x such that $|x|$ and $|l-x|$ are both large compared with 1. That this expectation is justified is confirmed in §5.

Finally, we have to supply an initial condition if the mathematical problem is to be properly posed. This should incorporate the expectation that, long before reversal, there is a quasi-steady temperature field everywhere. That is, for $t < -t_A$, where t_A is a dimensionless time much larger than any other parameter of the problem, the temperature field should take a quasi-steady form:

$$\theta = \theta_{qs} \quad \text{for} \quad t \leqslant -t_A.$$

However, if t_A is large enough, the temperature field over the heated region is independent of the initial field at all times such that $|t|$ is small compared with t_A. This is because the fluid still has a large negative velocity at $t = -t_A$; a fluid particle which is near the heated region at that time is convected away towards $x = -\infty$, and does not return until times close to $t = +t_A$, by which time everything interesting has already happened. Therefore the initial condition

$$\theta = 0 \quad \text{for} \quad t \leqslant -t_A \tag{2.5}$$

may be substituted with no loss of precision.

The problem to be solved, then, is defined by (2.2) or (2.4), together with boundary conditions (2.3a, b) and initial condition (2.5). We seek the temperature field $\theta\,(x, y, t)$. In fact, we are interested only in the boundary layer over the heated region, and therefore restrict attention to the region $0 < x < l$. Of particular interest is the heat transfer from the heated region, which is proportional to

$$Q(t) = l^{-\frac{2}{3}} \int_0^l -\theta_y|_{y=0}\, dx. \tag{2.6}$$

3. Approximate solution of the boundary-layer equation

We expect that long after reversal, i.e. as $t \to +\infty$, the temperature field will consist of a quasi-steady thermal boundary layer with origin at the new leading edge $x = 0$. 'Quasi-steady' means that the term θ_t in (2.4) is absent; in the region $0 < x < l$, θ must be a solution of

$$t\theta_x = \theta_{yy}$$

with boundary conditions

$$\theta = 1 \quad \text{on} \quad y = 0, \quad \theta \to 0 \quad \text{as} \quad y \to \infty. \tag{3.1}$$

The time t is only a parameter in this problem. The solution is

$$\theta = \operatorname{erfc}\,[\tfrac{1}{2}y(t/x)^{\frac{1}{2}}], \tag{3.2}$$

and we expect it to be valid for $t > t_2$, where t_2 is to be determined. We note that t_2 should be allowed to be a function of x, because this quasi-steady boundary layer is likely to be set up much sooner at points close to the leading edge ($x = 0$) than at points further downstream: the influence of the leading edge will first be felt at a point when fluid particles which have passed $x = 0$ arrive at the point.

The backwards-facing quasi-steady boundary layer, with leading edge at $x = l$, which we expect to be present long before reversal ($t \to -\infty$), can be analysed similarly. The temperature field in it is

$$\theta = \operatorname{erfc}\left[\frac{y}{2}\left(\frac{-t}{l-x}\right)^{\frac{1}{2}}\right], \tag{3.3}$$

and we expect this to be valid for $t < -t_1(x)$.

In between these two times, we expect the temperature to be given by a quasi-diffusive balance in which the term $t\theta_x$ in (2.4) is neglected. The equation becomes the diffusion equation:

$$\theta_t = \theta_{yy},$$

which must be solved subject to the same boundary conditions (3.1). An initial condition is also needed: this must be supplied by the requirement that the quasi-diffusive solution takes over, at time $t = -t_1$, from the backwards-facing quasi-steady boundary layer (3.3). Since t_1 depends on x, we expect the time at which the pure diffusion should be regarded as having started (the virtual origin of the diffusion) also to depend on x. The appropriate solution is therefore of the form

$$\theta = \operatorname{erfc}\,[\tfrac{1}{2}y/(t+t_0)^{\frac{1}{2}}], \tag{3.4}$$

where t_0 is a function of x. Equation (3.4) is expected to be the approximate solution for $-t_1 < t < t_2$.

We must now choose the quantities $t_0(x)$, $t_1(x)$ and $t_2(x)$. The choice of t_2 has been foreshadowed by the above remark that the influence of the leading edge $x = 0$ will be felt at a point when fluid particles which have passed $x = 0$ arrive at the point. Since flow reversal occurs at time $t = 0$, and the flow has constant acceleration, a particle at $x = 0$ will arrive at the point x on the heated region after a time $(2x)^{\frac{1}{2}}$. Now, we know in the case of the impulsively started flat plate (Hall 1969) that the change-over between the diffusive and the quasi-steady solutions takes place quickly, and begins when the influence of the leading edge is first felt. We therefore assume that the same thing happens here, and choose t_2 accordingly:

$$t_2(x) = (2x)^{\frac{1}{2}}. \tag{3.5}$$

The forwards-facing quasi-steady boundary layer is present after this time; the diffusive solution is valid before it.

The time $-t_1$ of change-over between the backwards-facing quasi-steady boundary layer and the diffusive solution is determined in a similar way. The diffusive solution will take over, at a given point, at a time when particles which have passed the leading edge $x = l$ first fail to arrive at the point before flow reversal causes them to be swept back towards $x = l$, the new trailing edge. This time is therefore given by

$$t_1(x) = [2(l-x)]^{\frac{1}{2}}. \tag{3.6}$$

The choice of $t_0\,(x)$ can now be made by requiring the temperature distribution to be continuous at $t = -t_1$, when the diffusive solution takes over from the backwards-facing quasi-steady boundary layer. This means that it cannot be made continuous at $t = t_2$, since there are no free parameters in the solution (3.2) for the forwards-facing quasi-steady boundary layer. However, this is to be expected, since that solution can depend only on upstream conditions, not on the downstream diffusion. The complete boundary-layer solution, given in §4, is also discontinuous at $t = t_2$. At $t = -t_1$, equations (3.3) and (3.4) are identical for all x (in the range $0 < x < l$) if t_0 is given by

$$t_0(x) = \tfrac{3}{2}[2(l-x)]^{\frac{1}{2}}; \tag{3.7}$$

we therefore choose this value of t_0.

We are fortunate in this problem because the functional forms of both the quasi-steady and the diffusive solutions are the same, and it is possible to make the temperature distribution continuous at $t = -t_1$. In the more realistic problems of a thermal boundary layer in a shear flow or of a viscous boundary layer, the functional forms are not the same. In those cases it is necessary to determine t_0 in some other way, for example, by requiring the flux of heat or mass along the boundary layer to be continuous at $t = -t_1$ for all x.

The dimensionless heat transfer $Q(t)$ from the heated region [see (2.6)] can now be computed from the solutions (3.2), (3.3) and (3.4), together with (3.5), (3.6) and (3.7). The results, when expressed in terms of a new dimensionless time $t' = tl^{-\frac{1}{2}}$, are

$$Q = \begin{cases} (2/\sqrt{\pi})\,(-t')^{\frac{1}{2}} \quad \text{for} \quad t' < -\sqrt{2}, \\ \dfrac{13\sqrt{2}}{27\sqrt{\pi}}\,(-t')^{\frac{3}{2}} + \dfrac{4}{9\sqrt{\pi}}\left(t' + \dfrac{3}{\sqrt{2}}\right)^{\frac{1}{2}}(\sqrt{2} - \tfrac{4}{3}t') \quad \text{for} \quad -\sqrt{2} < t' < 0, \\ \dfrac{1}{\sqrt{\pi}}\,t'^{\frac{3}{2}}\,(\sqrt{2} + \tfrac{16}{27}) + \dfrac{4}{9\sqrt{\pi}}\,[t' + \tfrac{3}{2}(2 - t'^2)^{\frac{1}{2}}]^{\frac{1}{2}}\,[(2 - t'^2)^{\frac{1}{2}} - \tfrac{4}{3}t'] \quad \text{for} \quad 0 < t' < \sqrt{2}, \\ (2/\sqrt{\pi})\,t'^{\frac{1}{2}} \quad \text{for} \quad t' > \sqrt{2}. \end{cases} \tag{3.8}$$

The fact that these results are independent of l, except in the definitions of t' and Q, indicates that a different non-dimensionalization would have removed l completely from the boundary-layer problem. Nevertheless, the present non-dimensionalization is more convenient for the exact solution of the full equation (2.2), performed in §5, and we therefore retain it.

4. Exact solution of the boundary-layer equations

Initially we shall show how the boundary-layer equation can be solved when the velocity is an arbitrary function of time, although for particular results it is always taken to be of the form (2.1). The governing equation for $\theta(x, y, t)$, a generalization of (2.4), is

$$\theta_t + u(t)\,\theta_x = \theta_{yy},$$

and the boundary and initial conditions are (2.3a, b) and (2.5). It is convenient to write the condition on $y = 0$ as

$$\theta(x, 0, t) = H(x) - H(l - x),$$

where $H(x)$ is the Heaviside step function. The equation can be reduced to the diffusion equation by the transformation

$$\xi = x - \int_0^t u(t')\,dt' \equiv x - s(t),$$

where $s(t)$ is the distance of a fluid particle from its position at $t = 0$. The function $\theta(\xi, y, t)$ satisfies the equation

$$\theta_t = \theta_{yy}$$

with boundary and initial conditions

$$\theta(\xi, \infty, t) = 0,$$
$$\theta(\xi, 0, t) = H[\xi + s(t)] - H[\xi - l + s(t)],$$
$$\theta(\xi, y, -t_A) = 0.$$

The problem now is of standard form. Its solution is

$$\theta(\xi, y, t) = \frac{2}{\sqrt{\pi}} \int_{y/2(t+t_A)^{\frac{1}{2}}}^{\infty} \left\{ H\left[\xi + s\left(t - \frac{y^2}{4\mu^2}\right)\right] - H\left[\xi - l + s\left(t - \frac{y^2}{4\mu^2}\right)\right]\right\} e^{-\mu^2}\,d\mu,$$

which for any smooth function s is equal to the sum of a number of error functions.

We now revert to the particular case of a constant acceleration with a single flow reversal at $t = 0$. In this case $s(t) = \frac{1}{2}t^2$, and with (x, y, t) as the independent variables instead of (ξ, y, t), the solution is

$$\theta(x, y, t) = F(x, y, t) - F(x - l, y, t), \tag{4.1}$$

where
$$F(x, y, t) = \frac{2}{\sqrt{\pi}} \int_{\gamma}^{\infty} H[x - \tfrac{1}{2}t^2 + \tfrac{1}{2}(t - y^2/4\mu^2)^2]\, e^{-\mu^2}\,d\mu; \tag{4.2}$$

here $\gamma = y/2(t + t_A)^{\frac{1}{2}}$. The integrand will be non-zero whenever the square bracket is positive, and to find all values of μ for which that is the case we must consider its zeros. They occur where

$$y^2/4\mu^2 = t \pm (t^2 - 2x)^{\frac{1}{2}},$$

and whenever $t^2 > 2x$ the last term is taken to be the positive square root. Note that in this analysis we must consider both positive and negative values of x, because of the second term in (4.1), as well as positive and negative values of t.

If *either* $t < 0$ and $x > 0$, *or* $t > 0$ and $x > \frac{1}{2}t^2$, then the square bracket in (4.2) has no zeros, and is positive throughout the range of integration. In those cases, therefore,

$$F(x, y, t) = \operatorname{erfc}\gamma.$$

However, when $t > 0$ and $0 < x < \frac{1}{2}t^2$, both zeros of the square bracket lie in the range of integration, and then

$$F(x, y, t) = \operatorname{erfc}\gamma - \operatorname{erfc}\{\tfrac{1}{2}y[t + (t^2 - 2x)^{\frac{1}{2}}]^{-\frac{1}{2}}\} + \operatorname{erfc}\{\tfrac{1}{2}y[t - (t^2 - 2x)^{\frac{1}{2}}]^{-\frac{1}{2}}\}.$$

Finally, when $x < 0$, for all t, there is just one zero in the range of integration, and then

$$F(x,y,t) = \operatorname{erfc}\gamma - \operatorname{erfc}\{\tfrac{1}{2}y[t+(t^2-2x)^{\frac{1}{2}}]^{-\frac{1}{2}}\}.$$

When all these results are put into (4.1), the solution for θ when x lies in the range $0 < x < l$, and t takes any value, is as follows. For $t > (2x)^{\frac{1}{2}}$,

$$\theta = \operatorname{erfc}\{\tfrac{1}{2}y[t+(t^2+2(l-x))^{\frac{1}{2}}]^{-\frac{1}{2}}\}. \tag{4.3}$$

For $t > (2x)^{\frac{1}{2}}$,

$$\theta = \operatorname{erfc}\{\tfrac{1}{2}y\,[t-(t^2-2x)^{\frac{1}{2}}]^{-\frac{1}{2}}\} - \operatorname{erfc}\{\tfrac{1}{2}y[t+(t^2-2x)^{\frac{1}{2}}]^{-\frac{1}{2}}\} + \operatorname{erfc}\{\tfrac{1}{2}y[t+(t^2+2(l-x))^{\frac{1}{2}}]^{-\frac{1}{2}}\}. \tag{4.4}$$

We may notice some interesting features of this solution.

(i) For large negative values of t, set $t = -t'$, where $t' \gg l^{\frac{1}{2}}$. Then the square bracket in (4.3) is equal to

$$-t'+t'\{1+2(l-x)/t'^2\}^{\frac{1}{2}},$$

which is approximately equal to $(l-x)/t'$. Equation (4.3) therefore reduces to that of the backwards-facing quasi-steady boundary layer, equation (3.3), as expected.

(ii) For large positive values of $t\,(\gg l^{\frac{1}{2}})$, the second and third terms in (4.4) cancel out to leading order, and together they are approximately equal to

$$[yl/4(2\pi)^{\frac{1}{2}}\,t]\,e^{-\frac{1}{8}y^2t^2},$$

whose maximum value (as y varies) is proportional to l/t^2, which is small. In the first term, we have

$$t-(t^2-2x)^{\frac{1}{2}} \approx x/t,$$

so that (4.4) reduces approximately to that of the forwards-facing quasi-steady boundary layer, equation (3.2), also as expected.

(iii) For very small values of t, equation (4.3) is applicable everywhere in $0 < x < l$ except where $x < \frac{1}{2}t^2$, which is a vanishingly small region and can be ignored. In this case, the square bracket in (4.3) is approximately equal to

$$t+[2(l-x)]^{\frac{1}{2}},$$

and therefore

$$\theta \approx \operatorname{erfc}\{\tfrac{1}{2}y[t+(2(l-x))^{\frac{1}{2}}]^{-\frac{1}{2}}\}. \tag{4.5}$$

This is of the form of the purely diffusive solution (3.4), but with t_0 equal to $[2(l-x)]^{\frac{1}{2}}$, rather than $\frac{3}{2}$ times that quantity as predicted in (3.7). This implies that the heat-transfer function Q has somewhat different values from those given by (3.8).

The heat-transfer function obtained from the complete solution (4.3) and (4.4) is as follows:

$$Q = \begin{cases} Q_1 \equiv (2/3\sqrt{\pi})\,[t'+(t'^2+2)^{\frac{1}{2}}]^{\frac{1}{2}}\,[(t'^2+2)^{\frac{1}{2}}-2t'] \quad \text{for} \quad t' < 0, \\ Q_2 \equiv Q_1 + (2/3\sqrt{\pi})\,(2t')^{\frac{3}{2}} \quad \text{for} \quad 0 < t' < \sqrt{2}, \\ Q_2 + (2/3\sqrt{\pi})\{[t'+(t'^2-2)^{\frac{1}{2}}]^{\frac{1}{2}}\,[(t'^2-2)^{\frac{1}{2}}-2t'] + [t'-(t'^2-2)^{\frac{1}{2}}]^{\frac{1}{2}} \\ \qquad \times [(t'^2-2)^{\frac{1}{2}}+2t']\} \quad \text{for} \quad t' > \sqrt{2}. \end{cases} \tag{4.6}$$

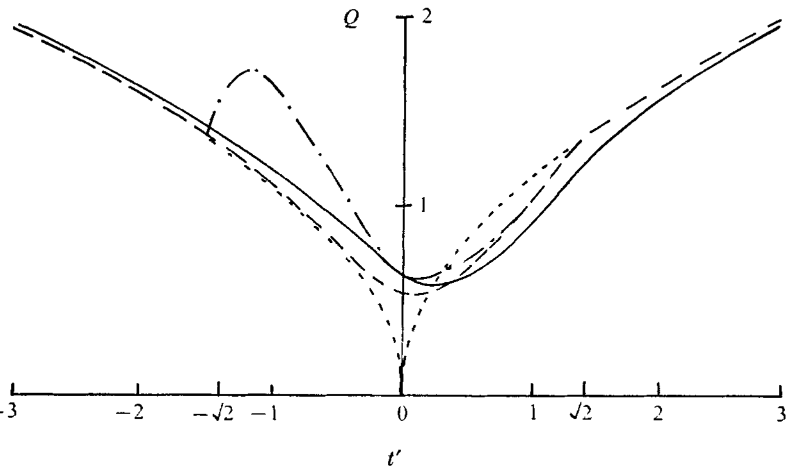

FIGURE 1. Graphs of the dimensionless heat transfer Q as a function of time t'. ———, exact solution (4.6); – – – –, approximate solution (3.8); —·——·—, approximate solution with $t_0 = [2(l-x)]^{\frac{1}{2}}$; - - - - -, quasi-steady solution.

The graph of Q against t' is plotted in figure 1, where this exact solution is compared with different approximations to it. The full curve is the exact solution given by (4.6). The broken curve is the approximate solution given by (3.8), incorporating into the diffusive part of the solution the value of t_0 given by (3.7). The agreement is really remarkably good, considering the sweeping nature of the approximations, with the error nowhere exceeding 17 %. This should be contrasted with the other approximate solution in which (4.5) describes the diffusive part, and which is represented by the dash-dot curve in figure 1. This solution, although more accurate near $t = 0$, is extremely inaccurate for a range of negative times, and would be unacceptable in practice. The reason for the inaccuracy lies in the fact that the thickness of the diffusive boundary layer, given by (4.5), is zero at $t = -[2(l-x)]^{\frac{1}{2}}$, which is just the time $-t_1$ at which it is supposed to take over from the backwards-facing quasi-steady boundary layer. There will therefore be gross discontinuities at this time, and the heat transfer per unit length, $-\theta_y|_{y=0}$, will locally become infinite. The former approximation, which ensures continuity at $t = -t_1$, is to be preferred. Also plotted on figure 1 is the grossest approximation of all, in which the temperature distribution is taken to be quasi-steady all the time, so that $Q = 2(|t'|/\pi)^{\frac{1}{2}}$ for all t'. This is very inaccurate, especially near $t' = 0$. Figure 1 demonstrates that the approximate solution developed in §3, with t_0 given by (3.7), is a good one, and suggests that it is justifiable to derive approximate solutions in more difficult problems by a similar method.

5. Exact solution of the full equation

In all the work so far we have assumed that the boundary-layer approximation can be applied throughout the period of flow reversal. That is, θ_x is small compared with θ_y everywhere at all times. However, θ is discontinuous at the ends of the heated strip, so θ_x is locally infinite, and as usual the boundary-layer approximation breaks down near those ends. We expect them to be surrounded by regions of length $O(1)$ in which the exact solution must be applied; but if l is large enough compared with 1, these regions will not influence the temperature over most of the strip. The boundary-layer solution given by (4.3) and (4.4) exhibits a further discontinuity, in θ_x, at the point $x = \frac{1}{2}t^2\,(t > 0)$, so that θ_{xx} is locally infinite there. Furthermore, the ratio of θ_x to θ_y in (4.4) is not small if $|x-\frac{1}{2}t^2|$ is small, even for large values of x and $l-x$. There is therefore a region surrounding $x = \frac{1}{2}t^2$ where the boundary-layer approximation breaks down. This is the point where, in the approximate solution of §3, the forwards-facing quasi-steady boundary layer takes over from the diffusing solution. In this section we solve the problem exactly, restricting attention mainly to the solution near $x = \frac{1}{2}t^2$ $(t > 0)$, and making an estimate of the error in the calculated heat transfer.

Equation (2.2) now governs the problem. We solve it, subject to the same boundary conditions as before, by taking Fourier transforms. Define

$$\hat{\theta}(k,y,t) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} e^{ikx}\,\theta(x,y,t)\,dx,$$

so that the transform of (2.2) is

$$\hat{\theta}_t + (k^2 - ikt)\,\hat{\theta} = \hat{\theta}_{yy}.$$

This can be converted into the diffusion equation by the substitution

$$\hat{\phi} = \exp\,(k^2t - \tfrac{1}{2}ikt^2)\,\hat{\theta}, \tag{5.1}$$

and $\hat{\phi}$ satisfies
$$\hat{\phi}_t = \hat{\phi}_{yy}.$$

The boundary and initial conditions are obtained from the transform of the corresponding, conditions on θ:

$$\hat{\phi}\,(k,y,-t_A) = \hat{\phi}(k,\infty,t) = 0,$$

$$\hat{\phi}(k,0,t) = \exp\,(k^2t - \tfrac{1}{2}ikt^2)\,\frac{i}{(2\pi)^{\frac{1}{2}}\,k}\,(1-e^{ikl}).$$

This problem, like that of §4, is in a standard form, and its solution can immediately be written down as

$$\hat{\phi} = \frac{i\sqrt{2}}{\pi k}\,(1-e^{ikl})\int_{\gamma}^{\infty} \exp\{-\tfrac{1}{2}ik(t-y^2/4\mu^2)^2 + k^2(t-y^2/4\mu^2)-\mu^2\}\,d\mu, \tag{5.2}$$

where $\gamma = y/2(t+t_A)^{\frac{1}{2}}$ again. When this is inserted into (5.1) it completes the solution for $\hat{\theta}$. The Fourier transform can now be inverted formally, and θ determined in the following form:

$$\theta(x,y,t) = F_1(x,y,t) - F_1(l-x,y,t), \tag{5.3}$$

where

$$F_1(x,y,t) = \frac{i}{\pi^{\frac{3}{2}}}\int_{-\infty}^{\infty} \exp\,(-ikx + \tfrac{1}{2}ikt^2 - k^2t)\,\frac{dk}{k}$$
$$\times\int_{\gamma}^{\infty} \exp\{-\tfrac{1}{2}ik(t-y^2/4\mu^2)^2 + k^2(t-y^2/4\mu^2)-\mu^2\}\,d\mu.$$

The essential feature of this equation is that the second integral is uniformly convergent for all real k, y and t. We can therefore reverse the order of integration, and write F_1 in the form

$$F_1 = \frac{i}{\pi^{\frac{3}{2}}}\int_\gamma^\infty e^{-\mu^2}\,d\mu \int_{-\infty}^\infty e^{-ikX-k^2Y}\,\frac{dk}{k}, \tag{5.4}$$

where

$$X = x - \frac{y^2t}{4\mu^2} + \frac{y^4}{32\mu^4}, \quad Y = \frac{y^2}{4\mu^2}.$$

The second integral in (5.4) is a familiar one, and if we define the inverse Fourier transform as a Cauchy principal value when the integrand has a pole on the real axis, the second integral is equal to

$$-i\pi\,\mathrm{erf}\,(X/2\surd Y).$$

Hence

$$F_1(x) = \pi^{-\frac{1}{2}}\int_\gamma^\infty \mathrm{erf}\left\{\frac{x - y^2t/4\mu^2 + y^4/32\mu^4}{y/\mu}\right\}e^{-\mu^2}\,d\mu. \tag{5.5}$$

This integral cannot be evaluated in closed form so we shall seek asymptotic formulae for values of $|x|$ much greater than 1 (negative values of x should be included because of the second term in (5.3)).

We notice first that, when the quantity in the curly brackets in (5.5) ($\bar{G}$, say) is large and positive, the error function is approximately equal to $+1$, and when it is large and negative the error function is approximately equal to -1. If erf{ } were replaced by sgn{ }, then the present solution would involve the sum of error functions, and would reduce to the boundary-layer solution of §4 (under the boundary-layer approximation, the $-k^2Y$ term in (5.4) is absent, and the second integral there would be equal to $-\pi\,\mathrm{sgn}\,X$). When y is positive, and $|x|$ large, the magnitude of $\bar{G}$ is large for both small and large values of μ. The exact theory differs from boundary-layer theory because, as it varies between γ (small) and ∞, μ may take a value which causes $\bar{G}$ to pass through zero, or close to it. Boundary-layer theory will be inaccurate if the range of values for which $|\bar{G}| \leqslant O(1)$ is large enough to affect the value of the integral. Alternatively, boundary-layer theory will be accurate if $\bar{G}$ passes through zero (or close to it) so rapidly that the value of the integral is scarcely affected.

We shall therefore estimate the difference J between the value of $F_1(x)$ as given by (5.5) and its value according to boundary-layer theory. Changing the variable from μ to λ/y, we obtain the following equation for J:

$$J = \frac{y}{\surd\pi}\int_{1/2(t+t_A)^{\frac{1}{2}}}^\infty [\mathrm{sgn}\,\{xG(\lambda)\} - \mathrm{erf}\,\{xG(\lambda)\}]\,e^{-\lambda^2y^2}\,d\lambda, \tag{5.6}$$

where

$$G(\lambda) = \lambda(1 - t/4x\lambda^2 + 1/32x\lambda^4). \tag{5.7}$$

The quantity in the square brackets is equal to erfc (xG) if $xG > 0$, and

$$-\mathrm{erfc}\,(-xG) \quad \text{if} \quad xG < 0;$$

it therefore falls off exponentially as xG takes values increasingly far from zero. The major contributions to the value of the integral when $|x|$ is large will arise from the vicinity either of zeros of $G(\lambda)$, or, if there are no zeros, of minima of

$|G(\lambda)|$, which we notice becomes infinite as $\lambda \to 0$ or $\lambda \to \infty$. The zeros of $G(\lambda)$ are given by

$$\lambda^2 = \lambda_{0\pm}^2 = (8x)^{\frac{1}{2}}[t \pm (t^2 - 2x)^{\frac{1}{2}}], \tag{5.8}$$

and the stationary points are given by

$$\lambda^2 = \lambda_{1\pm}^2 = (8x)^{-1}[-t \pm (t^2 + 6x)^{\frac{1}{2}}]. \tag{5.9}$$

In the range $0 < \lambda < \infty$, the function $G(\lambda)$ takes one of four different forms according to the values of x and t. These are shown in figure 2. If $x < 0$, the function takes one of the two forms shown in figure 2(a), and the integral will be dominated by the neighbourhood of the single zero λ_{0+}. For $x > 0$ and $t < (2x)^{\frac{1}{2}}$, G has no zeros (figure 2b) and the integral is dominated by the neighbourhood of the single minimum λ_{1+}. For $x > 0$ and $t > (2x)^{\frac{1}{2}}$, G has two zeros in the range (figure 2c) which will both contribute to the integral.

When $x < 0$ (figure 2a), an asymptotic expansion of the integral in inverse powers of $|x|$ can be obtained quite simply by considering the neighbourhood of the zero λ_{0+}. When $x > 0$ and $t > (2x)^{\frac{1}{2}}$ (figure 2c), and there are two zeros, a similar asymptotic expansion can be derived by considering each zero independently, provided that the zeros are well separated. However, we notice that, when t tends to $(2x)^{\frac{1}{2}}$, the two zeros $\lambda_{0\pm}$ and the minimum λ_{1+} all come together, so that the expansion procedure breaks down and another must be found. A different expansion can also be found when $x > 0$ and $t < (2x)^{\frac{1}{2}}$ (figure 2b), this time by considering the neighbourhood of the minimum λ_{1+}; this too breaks down when t tends to $(2x)^{\frac{1}{2}}$. In the following three subsections we derive the expansions for large positive values of x in the three cases (i) $t > (\surd 2x)^{\frac{1}{2}}$ and $|t-(2x)^{\frac{1}{2}}|$ not small, (ii) $t < (2x)^{\frac{1}{2}}$ and $|t-(2x)^{\frac{1}{2}}|$ not small, and (iii) $|t-(2x)^{\frac{1}{2}}|$ small. The method of expansion for $x < 0$ is the same as for one of the zeros in case (i), and is not considered further.

The lower limit of integration in (5.6) can without loss of accuracy be replaced by zero, since t_A is very large and t, in the cases of interest, is positive. An integration by parts then gives

$$J = J_2 \equiv \frac{x}{\surd\pi}\int_0^\infty \operatorname{erf}(y\lambda)\, G'(\lambda)\, e^{-x^2G^2(\lambda)}\, d\lambda \tag{5.10}$$

when $t < (2x)^{\frac{1}{2}}$ (G as in figure 2b), and

$$J = J_1 \equiv J_2 - \operatorname{erf}(y\lambda_{0+}) + \operatorname{erf}(y\lambda_{0-})$$

when $t > (2x)^{\frac{1}{2}}$ (G as in figure 2c). The corresponding corrections to the heat-transfer function are given by

$$\left.\frac{\partial J}{\partial y}\right|_{y=0} = q_J = \frac{2x}{\pi}\int_0^\infty \lambda G'(\lambda)\, e^{-x^2G^2(\lambda)}\, d\lambda \tag{5.11}$$

when $J = J_2$, with the additional term $-(2/\surd\pi)(\lambda_{0+}-\lambda_{0-})$ when $J = J_1$. The integrals in (5.10) and (5.11) can be further transformed by the change of variable $\xi = G(\lambda)$, but care must be taken with the inverse transformation in which λ is expressed in terms of ξ. The minimum value of $G(\lambda)$ is $\xi_1 = G(\lambda)$. The range of integration must be split into two parts, in the first of which ξ goes from ∞ to ξ_1, while λ goes from 0 to λ_1 and may be written as $\lambda = \lambda_-(\xi)$. In the second range

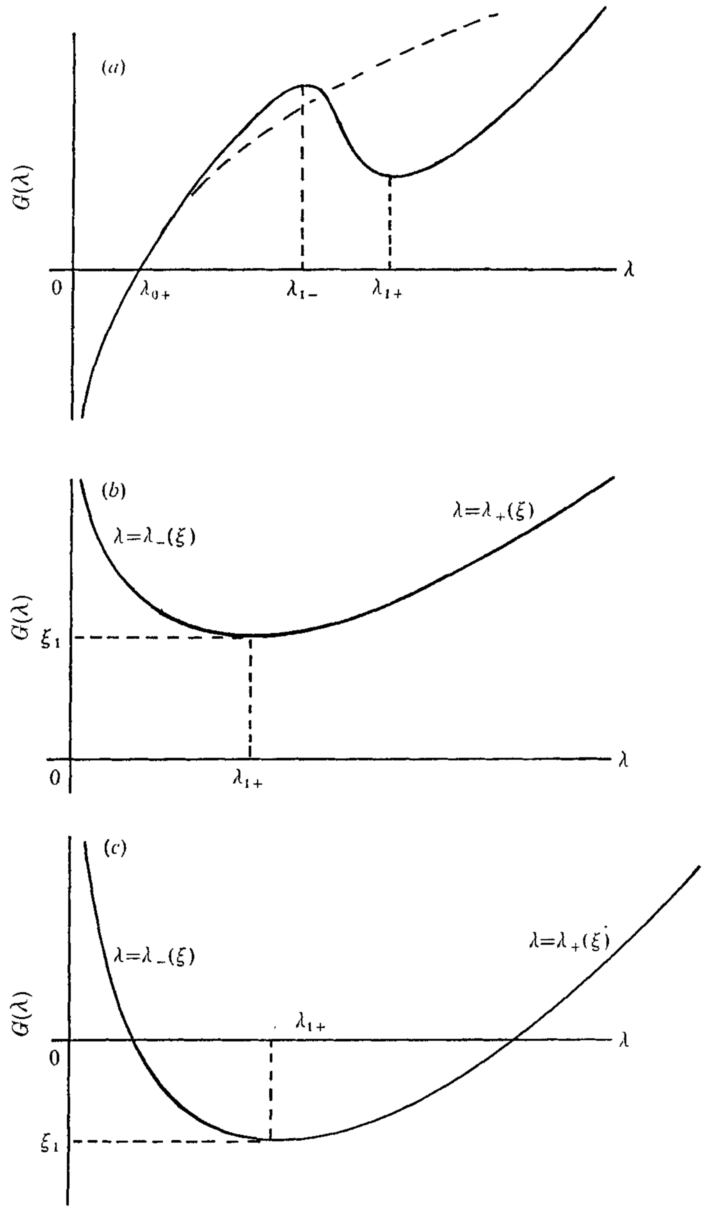

FIGURE 2. Graphs of $G(\lambda)$ against λ. (*a*) ——, $x < 0$, $t > (-6x)^{\frac{1}{2}}$ ($\lambda_{0-} < 0$); – – –, $x < 0$, $t < (-6x)^{\frac{1}{2}}$ ($\lambda_{0-} < 0, \lambda_{1\pm}$ complex). (*b*) $x > 0$, $t < (2x)^{\frac{1}{2}}$ ($\lambda_{1-} < 0$, $\lambda_{0\pm}$ complex). (*c*) $x > 0, t > (2x)^{\frac{1}{2}}$ ($\lambda_{1-} < 0$).

ξ goes from ξ_1 to ∞, while λ goes from λ_1 to ∞ and may be written as $\lambda = \lambda_+(\xi)$ (see figure 2*b*). Thus (5.11) becomes

$$q_J = \frac{2x}{\pi}\int_{\xi_1}^{\infty} [\lambda_+(\xi) - \lambda_-(\xi)]\, e^{-x^2\xi^2}\, d\xi \tag{5.12}$$

and the integral in (5.10) has the same form, but with $\lambda_\pm(\xi)$ replaced by

$$\tfrac{1}{2}[\pi \operatorname{erf}[y\lambda_\pm(\xi)].$$

(i) $t > (2x)^{\frac{1}{2}}$, $|t-(2x)^{\frac{1}{2}}|$ *not small*

In this case, $G(\lambda)$ has two distinct real zeros, and the value of ξ_1 is negative (figure 2*c*). The integrand is a maximum when $\xi = 0$, and to evaluate the integral asymptotically we expand the functions $\lambda_{\pm}(\xi)$ in powers of ξ. Writing

$$\lambda_{\pm}(\xi) = \lambda_{0\pm} + \alpha_{1\pm}\xi + \alpha_{2\pm}\xi^2 + \dots,$$

we use the definition (5.7) of $G(\lambda)$ and the values (5.8) of $\lambda_{0\pm}$ to obtain

$$\lambda_{0\pm} = \frac{(t \pm S)^{\frac{1}{2}}}{2(2x)^{\frac{1}{2}}}, \quad \alpha_{1\pm} = \pm\frac{t \pm S}{4S}, \quad \alpha_{2\pm} = \mp\frac{(2x)^{\frac{1}{2}}(2t \mp 3S)(t \pm S)^{\frac{3}{2}}}{16S^3},$$

where $S = (t^2-2x)^{\frac{1}{2}}$. When $x \gg 1$ and $t = O(x^{\frac{1}{2}})$ (that is, we do not consider values of t which are very large compared with the critical value $(2x)^{\frac{1}{2}}$), then $\lambda_{0\pm} = O(x^{-\frac{1}{4}})$, $\alpha_{1\pm} = O(1)$, $\alpha_{2\pm} = O(x^{\frac{1}{4}})$ and $\xi_1 = G(\lambda_1) = O(x^{-\frac{1}{4}})$. If we now set $\xi' = x\xi$, the lower limit of integration in (5.12) becomes $x\xi_1$, which is large and negative, and can be replaced by $-\infty$. The odd terms in the integrand are then seen to make no contribution to the integral, which for each zero becomes

$$\begin{aligned} q_{J\pm} &= \pm\frac{2x}{\sqrt{\pi}}\int_{-\infty}^{\infty}\left(\frac{\lambda_{0\pm}}{x} + \frac{\alpha_{2\pm}}{x^3}\xi'^2 + \dots\right)e^{-\xi'^2}\,d\xi' \\ &= \pm(2/\sqrt{\pi})\{\lambda_{0\pm} + \alpha_{2\pm}/2x^2 + O(x^{-\frac{13}{4}})\}. \end{aligned} \tag{5.13}$$

The total correction to the heat-transfer function in this case is

$$q_{J_2} = q_{J+} + q_{J-} - (2/\sqrt{\pi})(\lambda_{0+} - \lambda_{0-}),$$

so the leading terms in (5.13) cancel out, and we finally obtain

$$q_{J_2} = -[8(2\pi x)^{\frac{1}{2}}xS^3]^{-1}[(2t-3S)(t+S)^{\frac{3}{2}} + (2t+3S)(t-S)^{\frac{3}{2}}][1+O(x^{-\frac{3}{2}})], \tag{5.14}$$

which is $O(x^{-\frac{7}{4}})$. This is small compared with the boundary-layer value, which is $O(x^{-\frac{1}{4}})$. The correction to the temperature field is obtained in the same way, by expanding $\operatorname{erf}[y\lambda_{\pm}(\xi)]$ in powers of ξ.

The expansion breaks down when t is close to $(2x)^{\frac{1}{2}}$. To see this, set

$$t^2 = 2x(1+\epsilon), \quad \text{with} \quad \epsilon \ll 1,$$

so that $S = (2x\epsilon)^{\frac{1}{2}}$. The leading term $\lambda_{0+} - \lambda_{0-}$ in the expansion for $q_{J+} + q_{J-}$ is then proportional to $x^{\frac{1}{4}}\epsilon^{\frac{1}{2}}$, while the next term, $(\alpha_{2+} - \alpha_{2-})/x^2$, is proportional to $x^{-\frac{7}{4}}\epsilon^{-\frac{3}{2}}$, which becomes as large as the leading term when $\epsilon = O(x^{-\frac{3}{4}})$. Furthermore, $\xi_1 \propto \epsilon x^{-\frac{1}{4}}$, so $|x\xi_1|$ is no longer large when $\epsilon = O(x^{-\frac{3}{4}})$. Thus the whole expansion procedure ceases to be valid when ϵ is this small. We notice that, even then, the correction to the heat-transfer function is $O(x^{-\frac{5}{8}})$, which is still small compared with the boundary-layer value.

(ii) $t < (2x)^{\frac{1}{2}}$, $|t-(2x)^{\frac{1}{2}}|$ *not small*

In this case, $G(\lambda)$ has no zeros, and its minimum value ξ_1 is positive (figure 2*b*). The neighbourhood of this minimum will dominate the integral, so we expand $\lambda_{\pm}$ in powers of σ, where $\sigma = \xi - \xi_1$. Writing

$$\lambda_{\pm}(\xi) = \lambda_1 + \beta_{1\pm}\sigma^{\frac{1}{2}} + \beta_{2\pm}\sigma + \beta_{3\pm}\sigma^{\frac{3}{2}} + \dots, \tag{5.15}$$

we obtain

$$\lambda_1 = \frac{(S'-t)^{\frac{1}{2}}}{2(2x)^{\frac{1}{2}}}, \quad \beta_{1\pm} = \pm\frac{(S'-t)^{\frac{3}{2}}}{2(2x)^{\frac{1}{4}}S'^{\frac{1}{2}}} = \pm\beta, \quad \beta_{2+} = \beta_{2-},$$

$$\beta_{3\pm} = \pm\beta^{-1}\times O(1), \text{ etc.},$$

where $S' = (6x+t^2)^{\frac{1}{2}}$. Note that $\beta = O(x^{-\frac{1}{4}})$ and is positive. Thus $\lambda_+ - \lambda_-$ is an odd function of $\sigma^{\frac{1}{2}}$, and (5.12) becomes

$$\left.\begin{aligned} q_J &= \frac{4x}{\pi}\int_0^\infty [\beta\sigma^{\frac{1}{2}}+\beta_{3+}\sigma^{\frac{3}{2}}+\ldots]\, e^{-x^2(\xi_1+\sigma)^2}\, d\sigma \\ &\approx \frac{4x}{\pi}e^{-x^2\xi_1^2}\int_0^\infty [\beta\sigma^{\frac{1}{2}}+\beta_{3+}\sigma^{\frac{3}{2}}+\ldots][1-x^2\sigma^2+\ldots]\, e^{-2x^2\xi_1\sigma}\, d\sigma \end{aligned}\right\} \tag{5.16}$$

$$= \frac{\beta e^{-x^2\xi_1^2}}{(2\pi)^{\frac{1}{2}}x^2\xi_1^{\frac{3}{2}}}\left[1-\frac{15}{16x^2\xi_1^2}+\frac{3|\beta_{3\pm}|}{4\beta x^2\xi_1}+O(x^{-3})\right]. \tag{5.17}$$

In this expansion both the second and the third term in the square brackets are $O(x^{-\frac{2}{3}})$, while the value of q_J is $O(x^{-\frac{7}{4}}e^{-kx^{\frac{4}{3}}})$, where k is a constant. This is exponentially small, so that the boundary-layer solution is extremely accurate.

Once more, the expansion breaks down when $|t-(2x)^{\frac{1}{2}}|$ is small. If

$$t^2 = 2x(1-\epsilon), \quad \text{with} \quad \epsilon \ll 1,$$

ξ_1 is again proportional to $\epsilon x^{-\frac{1}{4}}$, so the second term in (5.17) is as large as the first when $\epsilon = O(x^{-\frac{3}{4}})$. This time, however, the values of S', β, etc., are not proportional to some power of ϵ, so the expansion (5.15) is valid however small ϵ is. The error lies in the expansion of $\exp[-x^2(2\xi_1\sigma+\sigma^2)]$, in which we assumed that $x^2\sigma^2$ was small when σ was such that $x^2\xi_1\sigma = O(1)$; these two terms are of equal magnitude when $\epsilon = O(x^{-\frac{3}{4}})$. When the expansion breaks down, $q_J = O(x^{-\frac{5}{4}})$ again.

(iii) $|t-(2x)^{\frac{1}{2}}| \ll 1$

Writing $t^2 = 2x(1-\epsilon)$ as in (ii), we consider values of $|\epsilon|$ much smaller than those for which the above expansions break down, i.e. $|\epsilon| \ll x^{-\frac{3}{4}}$. We can now modify the expansion procedure of (ii): in this case the integral (5.16) is dominated by values of σ such that $x\sigma = O(1)$, and $x^2\xi_1\sigma = O(x\xi_1) = O(\epsilon x^{\frac{3}{4}}) \ll 1$. We therefore have

$$\begin{aligned} q_J &= \frac{4x}{\pi}e^{-x^2\xi_1^2}\int_0^\infty [\beta\sigma^{\frac{1}{2}}+\beta_{3+}\sigma^{\frac{3}{2}}+\ldots][1-2x^2\xi_1\sigma+\ldots]\, e^{-x^2\sigma^2}\, d\sigma \\ &= \frac{2\beta\Gamma(\frac{3}{4})}{\pi x^{\frac{1}{2}}}e^{-x^2\xi_1^2}\left\{1+\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}\left(\frac{\beta_{3+}}{\beta x}-2x\xi_1\right)+\ldots\right\} \\ &= \frac{2\beta\Gamma(\frac{3}{4})}{\pi x^{\frac{1}{2}}}\{1+O[\max(x^{-\frac{1}{2}}, \epsilon x^{\frac{3}{4}})]\}, \end{aligned} \tag{5.18}$$

and here, too, $q_J = O(x^{-\frac{3}{4}})$. Note that this expansion is valid for $t > (2x)^{\frac{1}{2}}$ as well as $t < (2x)^{\frac{1}{2}}$; for values of ϵ so small that $|x\xi_1| \ll 1$, the distinction between the two cases is irrelevant, and manifests itself only because ξ_1 changes sign. The only regions for which we have not obtained a valid asymptotic expansion for q_J are those for which $|\epsilon| = O(x^{-\frac{3}{4}})$, where the expansions (5.14) and (5.17) both merge into (5.18). In such regions the solution can be expressed in terms of parabolic cylinder functions of argument $x\xi_1$ but this adds little to our understanding of the transition. Furthermore, all the above results indicate that q_J is never larger than $O(x^{-\frac{5}{8}})$, which is small compared with the boundary-layer value of $q = O(x^{-\frac{1}{4}})$. We therefore conclude that boundary-layer theory gives a good approximation to the heat-transfer function $q(x)$ for all values of x such that $0 < x < l$ and $|x| \gg 1$, $|l-x| \gg 1$.

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